+ **Задача 22.** *Вычислить первую и вторую производные по Фреше:*

$$J(x) = (\sum_{k=1}^{+\infty} x_k x_{k+1},\ x \in l^2.$$

Решение.

$$J(x) = \langle Ax, x \rangle,$$

$$Ax : x = (x_1, x_2, \dots, x_n, \dots) \to (x_2. x_3, \dots, x_{n+1} \dots), \quad A : l^2 \to l^2.$$

Оператор A является линейным и ограниченным. Найдем A^*:

очевидно

$$\langle Ax, y \rangle = x_2 y_1 + x_3 y_2 + \dots = x1 \cdot 0 + x_2 y_1 + \dots = \langle x, A^* y \rangle,$$

$$\|Ax\|_{\ell_2}^2 = \sum_{k=2}^{+\infty} x_k^2 \le \sum_{k=1}^{+\infty} x_k^2 = \|x\|_{\ell_2}^2$$

$$\underbrace{J'(x)}_{\mathcal{L}(\ell_2 \to \mathbb{R}^1)} \cong \underbrace{(A+A^*)x}_{\ell_2} = (x_2, x_3, x_4, \dots) + (0, x_1, x_2, \dots) =$$

$$= (x_2, x_1 + x_3, x_2 + x_4, \dots)$$

$$\underbrace{J''(x)}_{\mathcal{L}(\ell_2 \to \mathcal{L}(\ell_2 \to \mathbb{R}^1))} \cong \underbrace{A + A^*}_{\mathcal{L}(\ell_2 \to \ell_2)}$$

$J'(x)$: $J'(x)h = \langle (A+A^*)x, h \rangle_{\ell_2}$

$J''(x)$: $\underbrace{J''(x)h}_{\mathcal{L}(\ell_2 \to \mathbb{R}^1)} \cong \underbrace{(A+A^*)h}_{\ell_2}$, т.е. $(J''(x)h)f = \langle (A+A^*)h, f \rangle_{\ell_2}$ $\forall h, f \in \ell_2$

$$A^* : y = (y_1, y_2, \dots, y_n, \dots) \to (0, y_1, y_2, \dots, y^{n-1}, \dots).$$

$$J'(x)h = \langle (A + A^*)x, h \rangle, \quad J'(x) \in \mathcal{L}(l^2 \to \mathbb{R}^1),$$

$$(J''(x)h)f = \langle (A + A^*)h, f \rangle, \quad J''(x) \in \mathcal{L}(l^2 \to \mathcal{L}(l^2 \to \mathbb{R}^1)).$$

□

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$$J(u) = \int_0^{1/2} u^2(t)\,dt, \quad u \in L^2(0,1)$$

$$= \langle Au, u\rangle_{L^2(0,1)} \qquad A: u(t) \to \begin{cases} u(t), & [0, 1/2] \\ 0, & (1/2, 1] \end{cases}$$

$$= \|Au\|^2_{L^2(0,1)} \qquad A: \text{—}u\text{—}$$

A — лин., огр.

$$J'(u) \cong 2A^*Au$$

$$\langle Au, v\rangle_{L^2(0,1)} = \int_0^{1/2} u(t)\,v(t)\,dt = \langle u, A^*v\rangle_{L^2(0,1)}$$

$$A^* = A, \quad A^2 = A$$

$$(J'(u))h = \langle \underbrace{2A^*Au}_{2Au}, h\rangle_{L^2(0,1)} =$$

$$= 2\int_0^{1/2} u(t)\,h(t)\,dt$$

$$J''(u) \cong 2A^*A = 2A$$

$$((J''(u))h)g = \langle J''(u)h, g\rangle_{L^2(0,1)} =$$

$$= 2\int_0^{1/2} h(t)\,g(t)\,dt$$